Thus the control of Eq. (16) can be written as

$$u(t) = Ae^{Bt}b = e^{t}/\sinh\tau \tag{19}$$

where $A = [0 \quad -1]$, e^{Bt} is defined by Eq. (18), and $b = (0_g - 1/\sinh\tau)^T$. The analytic Fourier transform of $u(t)$ follows as

$$\bar{u}(\omega) = \left(\int_0^{\tau} e^{(1-i\omega)t}\mathrm{d}t\right)/\sinh\tau = [e^{(1-i\omega)\tau} - 1]/[(1-i\omega)\sinh\tau] \tag{20}$$

Since B is diagonalizable, we use the right and left eigenvector transformation method to solve Eq. (13), leading to

$$R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 2/\sqrt{2} \end{bmatrix}, \quad L = \begin{bmatrix} 2/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

from which it follows that

$$\beta = L^T b = -[1/(\sqrt{2}\sinh\tau), 1/(\sqrt{2}\sinh\tau)]^T$$

$$\gamma = (-1/[(1+i\omega)\sqrt{2}\sinh\tau], 1/[(1-i\omega)\sqrt{2}\sinh\tau])^T$$

$$A_1 = AR = [0, -2/\sqrt{2}], \; A_2 = Ae^{B\tau}R = [0, -2e^{\tau}/\sqrt{2}]$$

Thus, from Eq. (15) we have

$$\bar{u}(\omega) = [e^{(1-i\omega)\tau} - 1]/[(1-i\omega)\sinh\tau] \tag{21}$$

where Eq. (21) agrees with Eq. (20).

Conclusions

A computationally efficient algorithm has been presented for obtaining the complex Fourier transform of a class of vector functions that frequently occur in modern control theory. The basic algorithm requires 1) the evaluation of a single matrix exponential for the dynamics of the time-varying control; 2) the solution for either the right and left eigenvectors or a real Schur decomposition of the constant control dynamics matrix; 3) the sequential solution for the reducing subspace transformation vector p; and 4) the evaluation of a single scalar complex exponential.

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A Robust Compensator Design by Frequency-Shaped Estimation

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Introduction

MUCH has been made in recent literature[1-4] regarding the lack of robustness of servo loops designed by so-called 'optimal' stochastic control theory, i.e. Linear-Quadratic Synthesis (LQS) methods. It has even been claimed[5] that LQS should not be used on control systems for aerospace vehicles because of its sensitivity to plant uncertainties and its inappropriate bandwidth characteristics.

Frequency domain analysis of multivariable control loops indicates that linear state feedback provides good stability margins,[6] but that estimated state feedback can reduce these margins to zero.[1] It has been proposed to modify the Linear-Quadratic-Gaussian (LQG) estimator design procedure by adding fictitious process noise at the control inputs to restore the margins,[2,7] but that approach typically introduces high frequency modes into the estimator as the spectral density of this noise is increased.

This note describes an example using a frequency-shaped cost functional on measurement noise in LQG estimator design as a means for improving control loop robustness. The theory and implementation of frequency-shaped LQS design has been described elsewhere.[8-11] Kim[9,10] has shown that shaping the measurement noise of individual measurements using classical compensators, such as lead or lag networks, as shaping filters is effective in modifying multivariable controller robustness. This procedure results in transmission zeros being inserted into the estimator transfer matrix at prescribed frequencies. The selection of these frequencies (i.e. of the shaping filters) is accomplished by a frequency response analysis of the standard LQS controller design. The resulting estimator has higher order than the 'optimal' estimator, but it is no more complicated to design, and its bandwidth can be maintained in a reasonable range by choice of the usual LQS weighting parameters.

A Design Application

As an illustrative example of the effect of this procedure on frequency response of a multivariable loop, a controller design was undertaken on the single-input/two-output, 4th-order model for the azimuth pointing control servo of the Multiple Mirror Telescope.[12] This optical telescope, sponsored by the Smithsonian Institute and the University of Arizona, consists of six coaligned 1.8-meter primary mirrors in a hexagonal structure on an azimuth-altitude mount. The pointing controls include DC electric motors driving ring gears through a 100-to-1 reduction. Compliance in the gears combines with motor dynamics to produce the 4th-order model. A classically designed compensator with a bandwidth of about .5 Hz is presently used for control on each axis.

A state space representation of the azimuth axis system in the form

$$\dot{x} = Ax + Bu + Gw, \qquad z = Cx + v \tag{1}$$

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is given in Table 1. The control input is the commanded motor torque, and the measured outputs are motor shaft rate and azimuth pointing angle (relative to the earth). The transfer functions for these are

$$\frac{\omega_M(s)}{t_c(s)} = .278 \frac{(s+.00735)^2+(24.56)^2}{(s+.0432)\,[\,(s+.396)^2+(109.6)^2\,]} \tag{2a}$$

$$\frac{\theta_T(s)}{t_c(s)} = 1.675 \frac{1}{s(s+.0432)\,[(s+.396)^2+(109.6)^2\,]} \tag{2b}$$

Thus the system consists of low-frequency dynamics coupled to a high-frequency, lightly damped resonance. Since the resonance is well above the desired bandwidth of about .5 Hz, the approach taken to LQS controller design is to use a 2nd-order model of the system for estimator design, and then add frequency shaping to protect against the resonance that was not modeled. With multiple measurements, frequency shaping in the estimator must be done by multivariable analysis; however, the single input permits an assessment of the design by a classical frequency response with the loop broken at the input.

A reduced-order model in modal form (obtained by truncation) is given by

$$A_m = \begin{bmatrix} -.04324 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_m = \begin{bmatrix} -.3233 \\ -.3230 \end{bmatrix}$$

$$C_m = \begin{bmatrix} -.04320 & 0 \\ .00999 & -.0100 \end{bmatrix} \tag{3}$$

Applying the standard LQS technique to this 2nd-order model, with regulator output weighting only on θ_T, the following choice of weighting parameters (units as in Table 1)

$$Q_r = 10^8,\ R_r = 1,\ Q_e = 10^3,\ R_e = \mathrm{diag}\,(6\times 10^{-7},\ 6\times 10^{-12})$$

gives regulator poles at $s = -2.64 \pm 2.64j$ and estimator poles at $s = -4.14 \pm .94j$, with regulator and estimator gain matrices

$$K_r = [932.9 \quad -1000],\quad K_e = \begin{bmatrix} -122.3 & -14916. \\ -122.8 & -15197. \end{bmatrix}$$

When the resulting compensator is cascaded with the full-order system transfer functions of Eq. (2), the open-loop

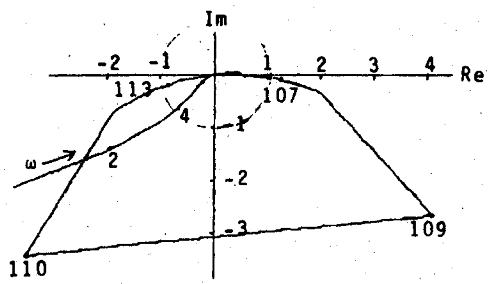

Fig. 1 Nyquist plot for reduced-order compensator.

Table 1 Multiple Mirror Telescope[a]

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.14\times 10^4 & -.833 & 1.14\times 10^6 & 0 \\ 0 & 0 & 0 & 1 \\ 6.03 & 0 & -603 & -1.47\times 10^{-3} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ .278 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.47\times 10^{-6} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

[a] $x = (\theta_M \omega_M \theta_T \omega_T)^T$, $z = (\omega_M \theta_T)^T$, $u = t_c$, $w = t_d$ where θ_M, ω_M are motor shaft angle and rate (in rad), θ_T, ω_T are telescope angle and rate (in rad), and t_c, t_d are control and disturbance torque (in lb-ft).

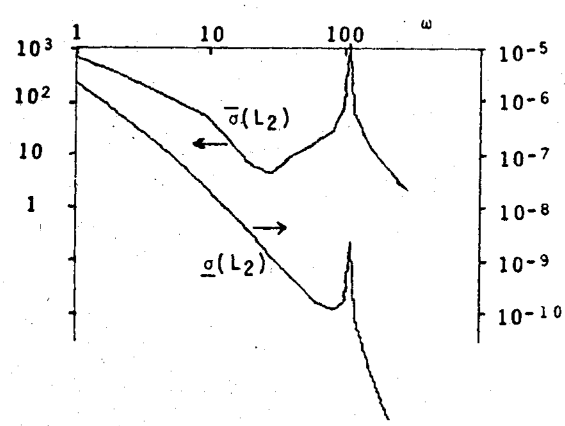

Fig. 2 Computed singular values of output return ratio matrix for reduced-order compensator.

Table 2 Open-loop transfer functions for Multiple Mirror Telescope controllers

a) Reduced-order compensator:

$$\frac{u(s)}{u'(s)} = 729.4 \frac{(s+1.926)(s+3.159)\,[(s-1.014)^2+(24.62)^2\,]}{s(s+.0432)(s+3.108)(s+10.41)\,[(s+.396)^2+(109.6)^2\,]}$$

b) Frequency-shaped θ_T:

$$\frac{u(s)}{u'(s)} = 738 \frac{(s+1.216)(s+18.24)\,[(s-.827)^2+(24.5)^2]\,[(s+77.4)^2+(77.6)^2)]}{s(s+.0432)\,[(s+6.43)^2+(11.5)^2]\,[(s+78.1)^2+(78.0)^2]\,[(s+.396)^2+(109.6)^2)]}$$

c) Frequency-shaped ω_M:

$$\frac{u(s)}{u'(s)} = 734 \frac{(s+1.832)(s+3.248)\,[(s-1.017)^2+(24.5)^2]\,[\,(s+.56)^2+(109.9)^2\,]}{s(s+.0432)(s+3.166)(s+10.9)\,[(s+77.4)^2+(77.4)^2]\,[(s+.396)^2+(109.6)^2\,]}$$

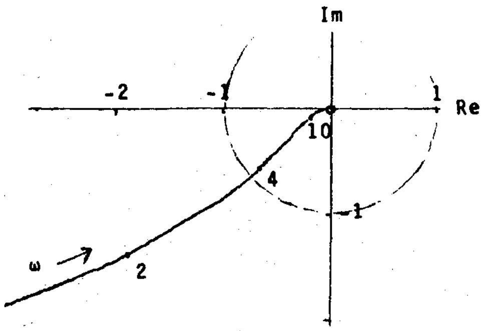

Fig. 3 Nyquist plot for frequency-shaped compensator.

transfer function (loop broken at input u) becomes that listed as a) in Table 2. Figure 1 presents a Nyquist plot of this transfer function, showing that the system is stable but with a phase margin of about 12 deg at the resonance.

If this loop is broken at the two outputs, a 2×2 output return ratio matrix,[10] L_2, can be formed using the above compensator. Figure 2 presents plots of the computed singular values of this matrix, showing amplification of the maximum singular value $\bar{\sigma}(L_2)$. The minimum singular value should be zero since L_2 has rank one. The resonance at 110 rad/s is evident. The singular vectors of L_2 at the resonant frequency show that $\bar{\sigma}(L_2)$ is most strongly influenced by output θ_T, while $\underline{\sigma}(L_2)$ is most strongly influenced by output ω_m.

The classical method for stabilizing a resonance to permit a wider control bandwidth is to introduce a notch filter into the loop, thus placing zeros at the resonant poles. This could be done at the input of this system, but it can also be accomplished with frequency shaping in the estimator by choosing a shaping function for one of the measurements of the form[11]

$$R^{-1/2}(s) = \frac{s^2 + 2\zeta\omega s + \omega^2}{s^2 + 2\zeta_d \omega s + \omega^2} \tag{4}$$

where ω is the resonant frequency, ζ is its damping ratio, and ζ_d is the desired damping. In this case, the selected values were $\omega = 110$ rad/s, $\zeta = .005$, and $\zeta_d = .707$; but the question is which measurement to filter?

Much of the literature on singular value analysis of multivariable control loops[13,14] seems to imply that robustness to model errors outside the desired bandwidth can be enhanced by reducing (or rolling off) $\bar{\sigma}(L_2)$ at high frequency. This suggests that Eq. (4) should be applied to θ_T which effects $\bar{\sigma}(L_2)$. Implementing such an estimator, by combining a realization of Eq. (4) with (3) and using the same weighting as above, gives an additional set of estimator poles at $s = -77.8 \pm 77.8j$, with gain matrices[9,11]

$$[K_e] = \begin{bmatrix} -123.4 & -13970 \\ -123.9 & -14254 \end{bmatrix}$$

$$[K_v] = \begin{bmatrix} 4.96 \times 10^{-7} & 2.40 \times 10^{-4} \\ 4.12 \times 10^{-6} & 4.99 \times 10^{-4} \end{bmatrix}$$

When combined with K_r to form a 4th-order compensator, this estimator indeed substantially reduces the resonance in $\bar{\sigma}(L_2)$ of Fig. 2, while leaving $\underline{\sigma}(L_2)$ unaffected. However, the open-loop transfer function becomes that listed as (b) in Table 2, whose Nyquist plot is essentially the same as Fig. 1. Thus, no margin improvement occurs for the control input.

If the same notch filter in Eq. (4) is applied to ω_M, which is the standard rate feedback signal for increased damping, the same estimator poles occur, but the gain matrices become

$$[K_e] = \begin{bmatrix} -120.9 & -15922 \\ -121.5 & -16208 \end{bmatrix}$$

$$[K_v] = \begin{bmatrix} 4.15 \times 10^{-4} & .0567 \\ .00125 & .0239 \end{bmatrix}$$

When this estimator is combined with K_r, the resulting compensator leaves $\bar{\sigma}(L_2)$ of Fig. 2 unaffected, and the open-loop transfer function becomes that listed as (c) in Table 2. Thus the zeros of Eq. (4) appear in the loop and the resonance is gain stabilized, as seen in the Nyquist plot in Fig. 3. The bandwidth of this system is about .5 Hz, but it has sufficient margin to permit expansion to as much as 1.5 Hz.

Conclusions

Frequency shaping the measurement noise in Linear-Quadratic-Gaussian (LQG) estimator design, using classical single-variable filters, is an easy and effective means for improving the frequency response of some Linear-Quadratic-Synthesis (LQS) controllers. This seems particularly true when phase lead is needed since the method permits placing of estimator transmission zeros. However, as seen in the example, the method applies to plant outputs, but for multivariable loops, the effect on input characteristics is not always clear. Since the LQS regulator and estimator are mathematical duals, a similar procedure on input frequency shaping by regulator design is also possible, and a combined procedure is currently being investigated.

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Book Announcements

FLETCHER, R., University of Dundee, ***Practical Methods of Optimization: Volume 1, Unconstrained Optimization.*** John Wiley and Sons, New York, 1980, 120 pages. $31.95.

Purpose: This book aims to present those aspects of optimization methods which are currently of foremost importance in solving real life problems. Theory, numerical implementation, and experimentation are included. This book, or parts of it, have been used at both the undergraduate and graduate levels.

Contents: Introduction. Structure of methods. Newton-like methods. Conjugate-direction methods. Restricted-step methods. Sums of squares and non-linear equations. References. Index.

FLETCHER, R., University of Dundee, ***Practical Methods of Optimization: Volume 2, Constrained Optimization.*** John Wiley and Sons, New York, 1981, 224 pages. $34.95.

Purpose: Same as for Volume 1.

Contents: Introduction. Linear programming. Theory of constrained optimization. Quadratic programming. General linearly constrained optimization. Nonlinear programming. Other optimization problems. Non-differentiable optimization. References. Index.

VINH, N.X., University of Michigan, ***Optimal Trajectories in Atmospheric Flight.*** Elsevier Science Publishing Co., New York, 1981, 402 pages. $85.00.

Purpose: Even though it is based on lecture notes for a graduate course on trajectory optimization, this book is designed primarily as a reference text. The book is self-contained in that optimization theory is presented prior to applications to aircraft trajectories and atmospheric spacecraft trajectories.

Contents: Introduction. Optimization theory. Switching theory. Equations of motion. Aerodynamics and propulsive forces. General properties of optimal trajectories. Flight in a horizontal plane. Optimal coasting flight. Supersonic cruise. Supersonic turn. Supersonic maneuvers in a vertical plane. Energy state approximation. Modified Chapman's formulation for optimal re-entry trajectories. Optimal planar re-entry trajectories. Optimal glide of re-entry vehicles. Orbital aerodynamic maneuvers. Index.

MEES, A.I., Cambridge University, ***Dynamics of Feedback Systems.*** John Wiley and Sons, New York, 1981, 214 pages. $44.95.

Purpose: This book is directed toward control engineers and applied mathematicians. It has been written so that beginning graduate students can understand the material.

Contents: Introduction. Qualitative theory of ordinary differential equations. The feedback system viewpoint. Stability of feedback systems. Periodic solutions and the method of harmonic balance. The Hopf bifurcation. References. Index.

BARNETT, S., University of Bradford, ***Matrices in Control Theory.*** Robert E. Krieger Publishing Company, Melbourne, Fla., 1984, 192 pages. $14.50.

Purpose: This is a revised edition of the 1971 text. Some of the content of the book has been changed to sharpen the emphasis to the control field. This book is intended as a reference text for engineers, scientists, and applied mathematicians working in control theory and related areas involving applications of matrices.

Contents: Polynomial matrices. Polynomials. Rational matrices. Stability and inertia. Matrix Riccati equations. Generalized inverses. Unimodal matrices. Appendices. Index.

GOODWIN, G.C., University of Newcastle, and **SIN, K.S.,** ***Digital Equipment, Adaptive Filtering Prediction and Control.*** Prentice-Hall Inc., Englewood Cliffs, N.J., 1984, 540 pages. $41.95.

Purpose: This book presents the theory of adaptive filtering, prediction, and control in a unified fashion. The emphasis is primarily on linear discrete-time systems because of the importance of digital computers in the practical application of the theory. The book is aimed at senior undergraduate students, graduate students, and researchers. Relevant background material is summarized in the appendices in order to make the book self-contained.

Contents: Introduction to adaptive techniques. Models for deterministic dynamical systems. Parameter estimation for deterministic systems. Deterministic adaptive prediction. Control of linear deterministic systems. Adaptive control of linear deterministic systems. Optimal filtering and prediction. Parameter estimation for stochastic dynamic systems. Adaptive filtering and prediction. Adaptive control of stochastic systems. Appendices. References. Index.